Maximally Allowed Perturbation for Stable Motion using MATLAB

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Motivation

The motivative question is why Earth can stay on its orbit despite of a perturbation caused by Jupiter. We see everyday that situations get quickly out of control due to a tiny error. Indeed, it seems very easy to disturb and destroy certain systems, just by disturbing them. Even with a simple system, for example, if you quickly vibrate an oscillating pendulum, or if you try to bounce a pin pong ball on a racket[5], a system will start exhibiting a variety of chaotic outputs (e.g. frequency of a pendulum, height of a pin pong ball), even though the input is a tiny force. Naturally, one might say, "everything which is not forbidden is allowed," and we are led to suppose that any system with some perturbation will eventually realize all the possible behaviors it could realize after sufficiently long time (Ergodic Hypothesis). Nevertheless, we see many systems that are not chaotic, and even with a perturbation, do not experience *every* configuration that is classically possible. Here, we shall examine a condition in which a dynamical system can be stable and periodic using MATLAB.

Theory

Suppose we have a Hamiltonian,

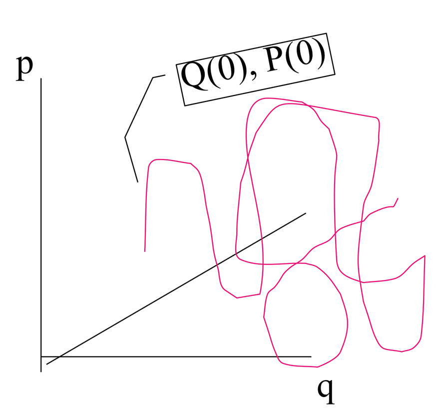
$$\mathcal{H}(p,q)$$

Now, consider a generator F for a canonical transformation such that

$$K = H + \frac{\partial F}{\partial t}$$

If $K = 0$,

$$\dot{Q} = \frac{\partial K}{\partial P} = 0\ ,\ \dot{P} = \frac{\partial K}{\partial Q} = 0$$



In the same sprit, consider a canonical transformation $S(\vec{q},\vec{J})$ such that

$\vec{q},\vec{p} = \frac{\partial S(\vec{q},\vec{J})}{\partial \vec{q}} \rightarrow \vec{J},\vec{\theta} = \frac{\partial S(\vec{q},\vec{J})}{\partial \vec{J}}$ where $S(\vec{q},\vec{J})$ is a solution of *Hamilton-Jacobi* equation $\mathcal{H}[\vec{q}, \frac{\partial S(\vec{q},\vec{J})}{\partial \vec{q}}] = \mathcal{H}(\vec{J})$ where $S(\vec{q},\vec{J})$ is the generator of canonical transformation.

Now, we consider a poison bracket $\{I_i, I_j\}$ where $i \neq j$.

Here, Hamiltonian H is conserved, and since $\frac{dI}{dt} = \{I, H\} + \frac{\partial I}{\partial t}$ is an evolution equation, $\{I_i, I_j\}$=0 implies that I_i, I_j…form planes (in two dimension) that *foliate* in a phase space. If there are n such I_i exist for Hamiltonian $\mathcal{H}$ with n degrees of freedom, such a system is called *integrable*.

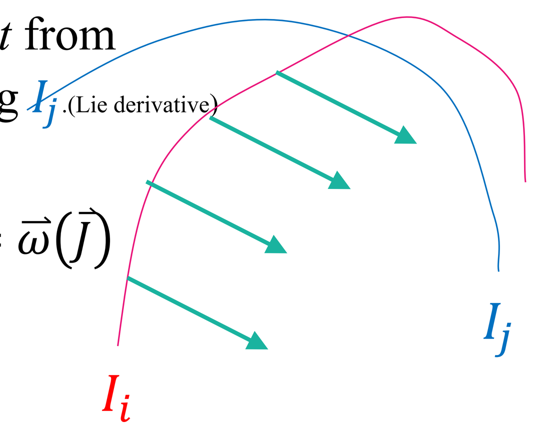
$\{I_i, I_j\}$=0 → I_i does not *stick out* from the surface as it is dragged along I_j (Lie derivative)

Generally, we can construct,

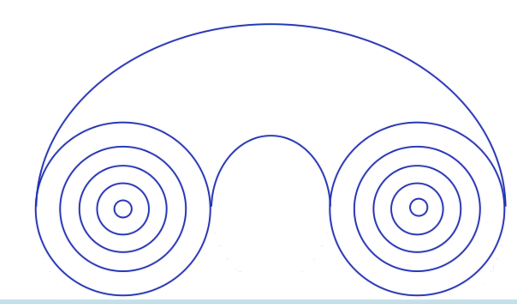
$\dot{\vec{J}} = -\frac{\partial \mathcal{H}(\vec{J})}{\partial \vec{\theta}} = 0,\ \dot{\vec{J}} = \frac{\partial \mathcal{H}(\vec{J})}{\partial \vec{J}} = \vec{\omega}(\vec{J})$

which gives,

$\vec{J}$ =constant, $\vec{\theta} = \vec{\omega}\cdot t + \vec{\theta}_0$



Then, $\vec{\theta}$ periodic and will form a n-dimensional *tori* for Hamiltonian with n-degrees of freedom. The motions lie on the surfaces of constant $\vec{J}$.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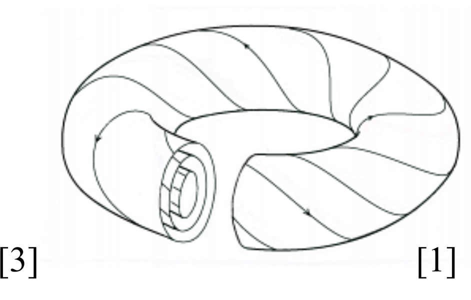
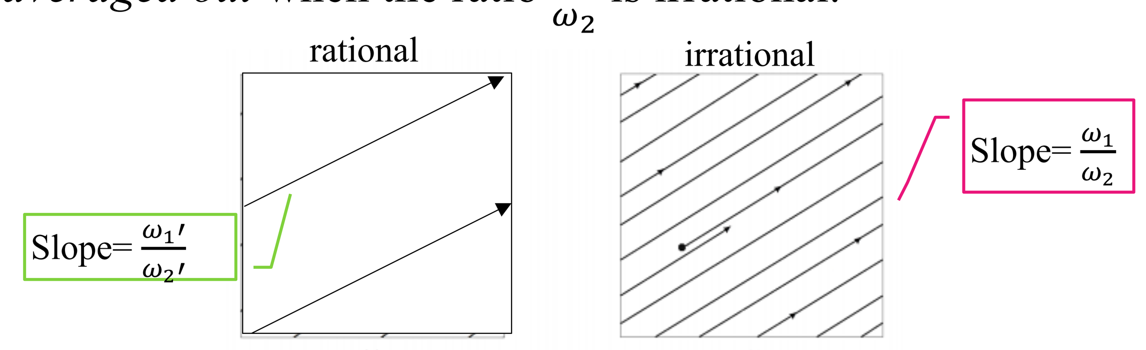

[1]

Consider a perturbation H_1, thus, the new Hamiltonian is,

$$H = H_0(J) + \varepsilon H_1(J,\theta)$$

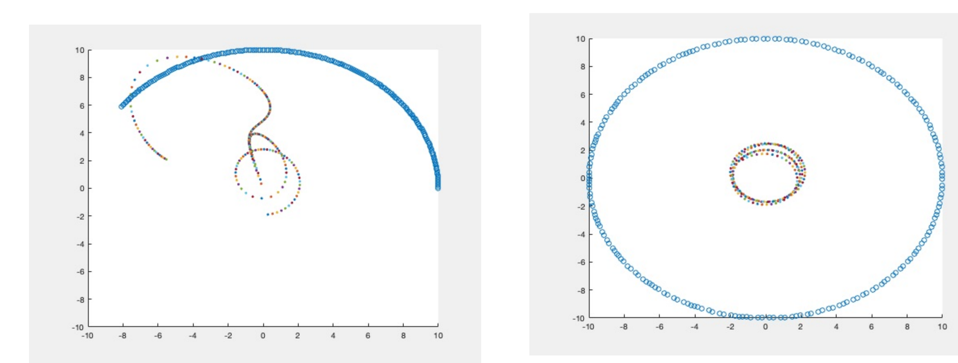
where ε is a small dimensionless parameter.

Can we make a canonical transformation that the resulting H only depends on J. We search for a generating function S which satisfies Hamilton Jacobi equation $H\left[\frac{\partial S}{\partial \vec{\theta},}\ \vec{\theta}\right] = H(\vec{J})$. Writing the generating function S as $S(\vec{J},\vec{\theta}) = \vec{\theta}\cdot\vec{J} + \varepsilon S_1(\vec{J},\vec{\theta})$ where $\vec{\theta}\cdot\vec{J}$ is the identity transformation, we have $\vec{\omega}\cdot\frac{\partial S_1(\vec{J},\vec{\theta})}{\partial\vec{\theta}} = -H_1(\vec{J},\vec{\theta})$. By letting $H_1(\vec{J},\vec{\theta}) = \sum_k H_k e^{ik\theta}$, we get as a solution, $S_1 = i\sum_k \frac{H_k}{k\cdot\omega_k} e^{ik\theta}$. Here, we see that S_1 diverges if $k\cdot\omega_k$= 0 for some $k \in \mathbb{N}$. Such ω_k is called *resonant* and rational. In fact, if a system is resonant, it gets destroyed by the slightest perturbation. It can be thought that the perturbation is smoothed out, *averaged out* when the ratio $\frac{\omega_1}{\omega_2}$ is irrational.



The *strength* of irrationality here can be defined as how large the natural numbers must be to form a fraction that approximates the number. e.g. $\frac{7}{13}$ is more irrational than $\frac{1}{2}$.. In fact, the most irrational number turns out to be the golden ratio: $\frac{1+\sqrt{5}}{2}$ which is the frequency that is most resistant to the perturbation. Earth's orbit with a smaller and larger frequency of "Jupiter" is shown in the next page.

MATLAB



Earth's orbit with a smaller and larger frequency of "Jupiter"

Qualitatively, you can see that the direction of gravitational force acting on Earth is being averaged out.

Now, let's try to examine more general dynamical systems. Consider the *Hénon-Heiles* Hamiltonian that describes the motion of stars around a galactic center, assuming the motion is restricted to the *xy* plane;

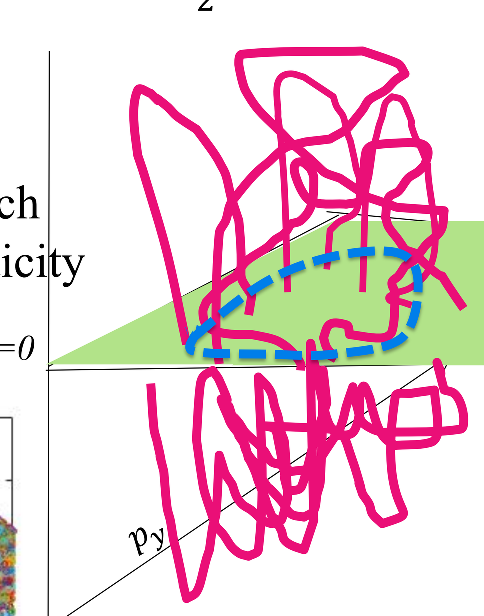
$$H = \frac{1}{2}{p_x}^2 + \frac{1}{2}x^2 + \frac{1}{2}{p_y}^2 + \frac{1}{2}y^2 + \left[x^2 y - \frac{1}{3}y^3\right]$$

where $H(x,y) = \frac{1}{2}{p_x}^2 + \frac{1}{2}x^2 + \frac{1}{2}{p_y}^2 + \frac{1}{2}y^2$ is a harmonic term and $V(x,y) = \left[x^2 y - \frac{1}{3}y^3\right]$ is a nonlinear perturbation term. It can be shown that in action-angle coordinate, $H(\theta_1,\theta_2,J_1,J_2) = H_0(J_1,J_2) + \epsilon H_1(\vec{\theta},\vec{J})$ where $H_0 = \omega_1\cdot J_1 + \omega_2\cdot J_2$ and $H_1 = x^2 y - \frac{1}{3}y^3$.

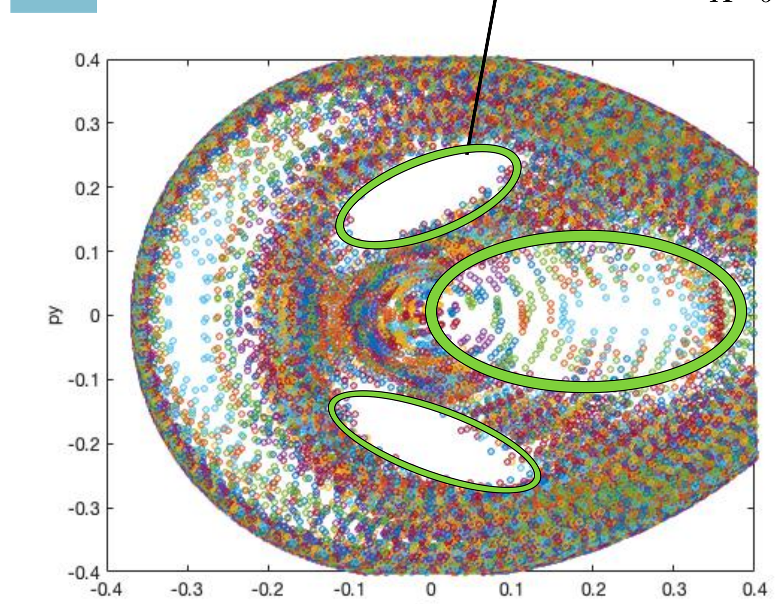
We shall examine Poincare sections in the $p_y - y$ plane, so we set x=0 and find the initial energy, $E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}y^2 - \frac{1}{3}y^3$.

When the energy $E = \frac{1}{12}$,

There exist multiples of tori, which indicates that stability and periodicity of motion.

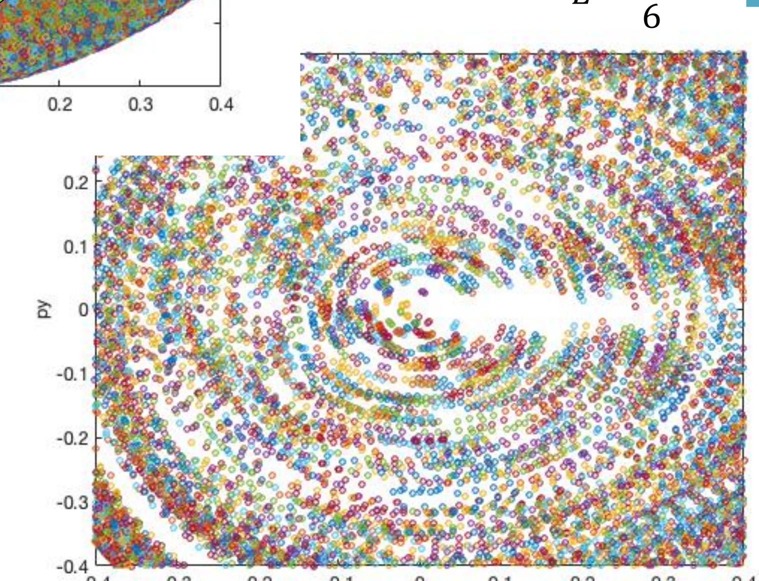


Poincare section; $p_y - y$ plane



$E = \frac{1}{12}$

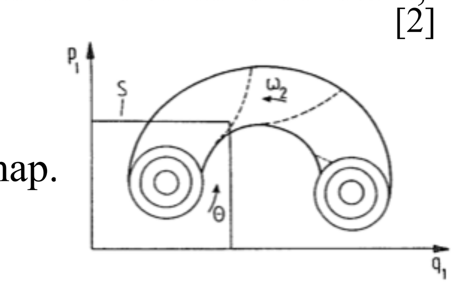
As you increase the Energy, the tori have been destroyed. At $E\frac{1}{6}$, phase space is filled with chaos.



$E = \frac{1}{6}$

Remarkably, we can conjure up a usual map, *standard map (or twist map)*. In figure in the right, since the radius of the torus is constant, $r_{i+1} = r_i$. And, since θ $(= \omega_1\cdot t + \delta)$ shifts by $\omega_1\cdot T$ where $T = \frac{2\pi}{\omega_2}$,

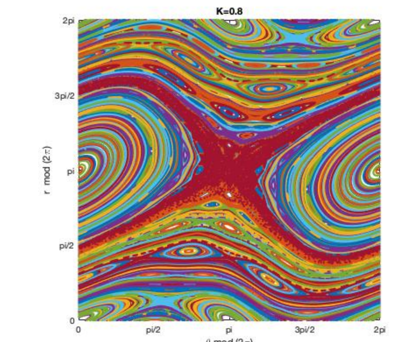
$\theta_{i+1} = \theta_i + 2\pi\frac{\omega_1}{\omega_2}$. And this defines the map.

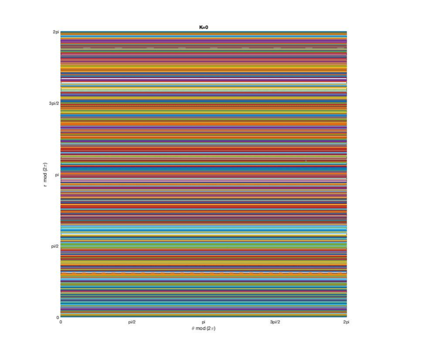


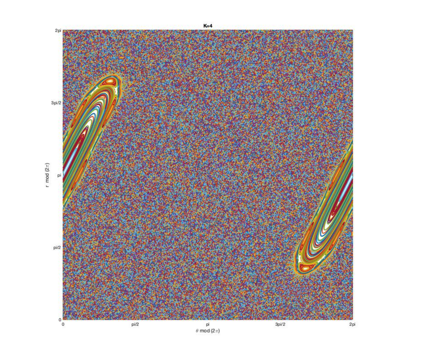
If we now perturb H_0, assuming that the perturbation is periodic and since $\frac{\omega_1}{\omega_2}$ is only depends on r, with a scaling, we have,

$$r_{i+1} = r_i. + \ \varepsilon\cdot sin(\theta_i)$$
$$\theta_{i+1} = \theta_i + r_i\ (\text{mod } 2\pi)$$

This map works in general, *such as Hénon-Heiles* system as well as Three-body problem. applying this map.





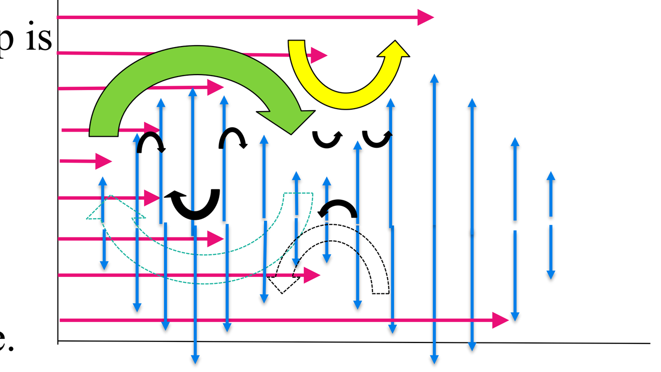


In addition to the fact that the tori break down as the perturbation increases, we can see the stability of tori as a function of the ratio of two frequencies.

Also, you can see that the map is filled with elliptic curves and hyperbolic curves.

Since this process repeats and makes smaller curves each time tori breaks down.

You can see a fractal structure.



Here, the concept of stability in its generality is examined by reviewing an integrability, mainly on the action-angle coordinate. Then, a few dynamical systems are examined. We saw that a system, with an appropriate condition, can maintain its stability even after some perturbation turns on and do not become chaotic immediately.

References

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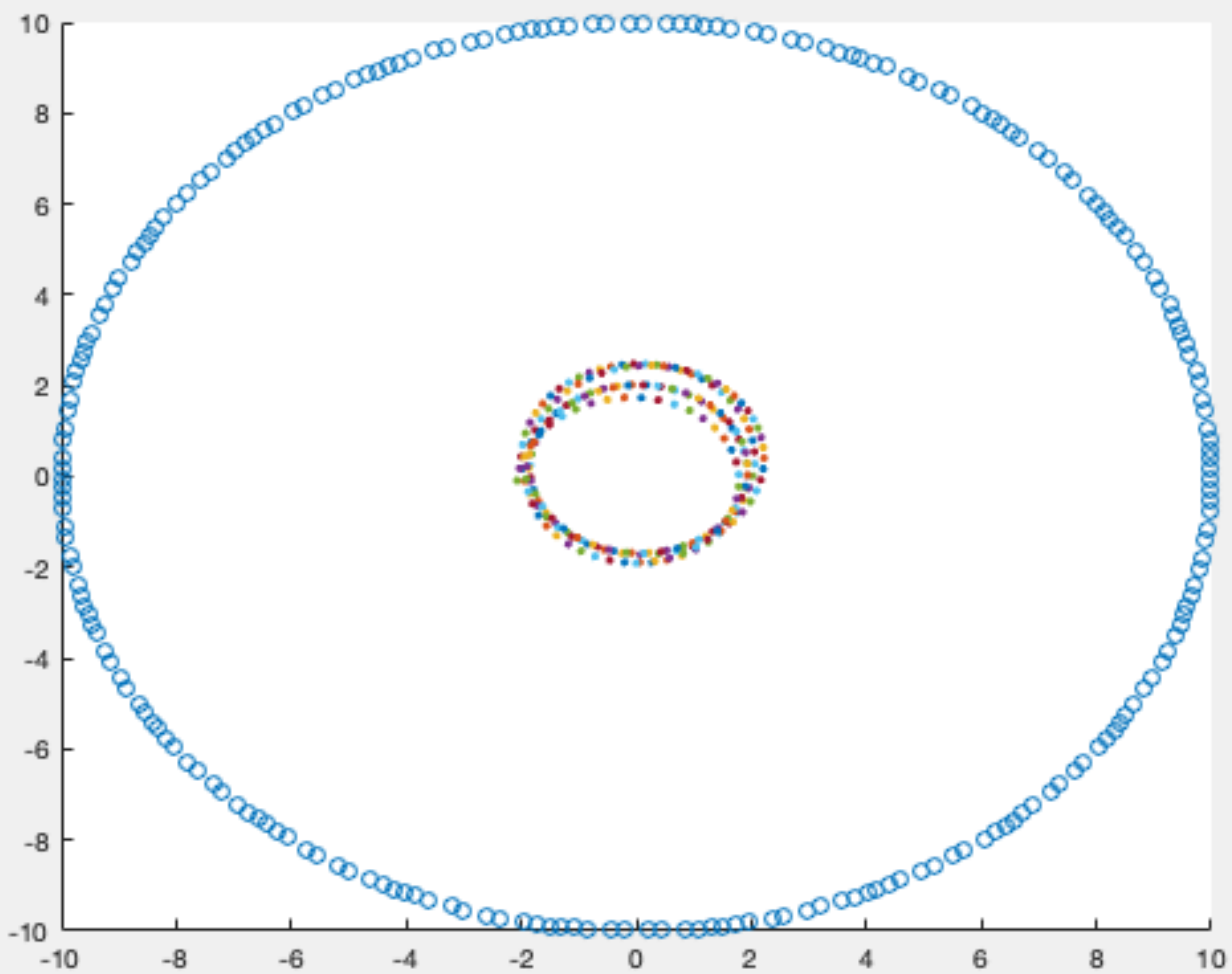
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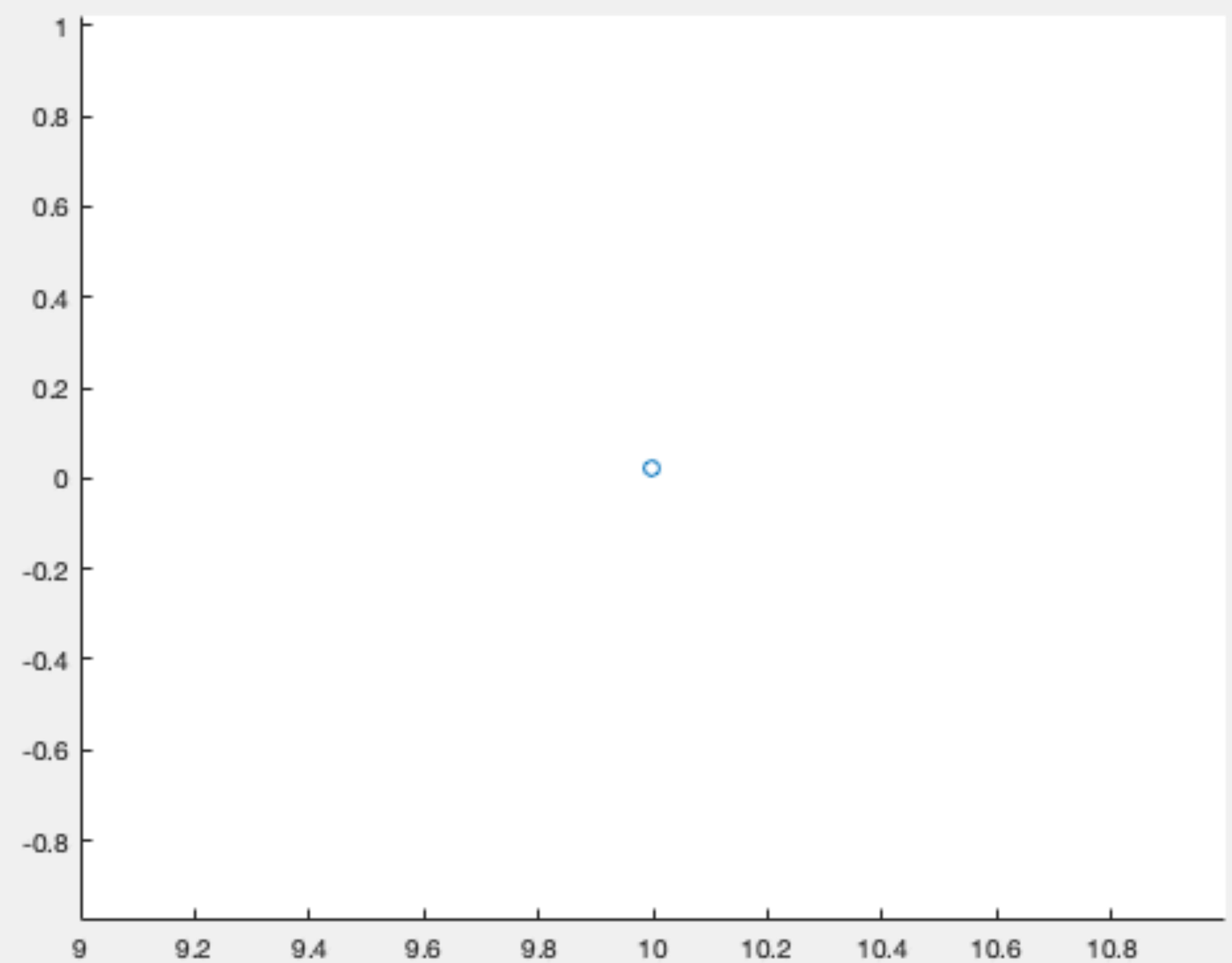
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